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Rearrangements of a conditionally convergent series summing to logarithms of natural numbers

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It is a counterintuitive idea to calculus students that conditionally convergent series may be rearranged to converge to different sums. Some nice examples can be helpful and fascinating to students. This note describes a family of such rearrangements suitable for calculus and undergraduate analysis students. There is one series for each $k \in \mathbb{N}$, with $k = 1$ the original series. The cases of $k = 1$ and $k = 2$ are separately presented, as they are particularly easy to show to a calculus class. The general case uses material from a first calculus class but is more involved. Various versions of the series are known and, for $k > 1$, wonderful.

For each series (labeled $k \in \mathbb{N}$) the positive terms and the negative terms form two harmonic series. The order of the two series are preserved so it is easy to see they are rearrangements.

A simple series, k=1. The original series is:

$$1 - 1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n} + \ldots. \tag{1}$$

The positive terms and the negative terms form two harmonic series. The partial sums are $s_n = 0$ for n even and $s_n = \frac{2}{n+1}$ for n odd. The series converges to $0 = \ln(1)$.

First rearrangement, k=2. The first rearrangement is

$$1 + \frac{1}{2} - 1 + \frac{1}{3} + \frac{1}{4} - \frac{1}{2} + \frac{1}{5} + \frac{1}{6} - \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{n} + \ldots. \tag{2}$$

To see convergence, we compare the series (2) to the alternating harmonic series,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots$$

and denote the n-th partial sum of the alternating harmonic series as h_n. Observe that in series (2), every third term plus its predecessor is $\frac{1}{2n} - \frac{1}{n} = -\frac{1}{2n}$. That contraction of the $3m-1$-th and $3m$-th terms converts series (2) into an alternating harmonic series and shows that its partial sums satisfy $s_{3m} = h_{2m}$. Calculating the other partial sums s_n for series (2) yields

$$s_n = \begin{cases} h_{\frac{2n}{3}} & \text{if } n = 3m, \\ h_{\frac{2(n+1)}{3}-1} + \frac{1}{2\left(\frac{n+1}{3}\right)} & \text{if } n = 3m-1, \\ h_{\frac{2(n+2)}{3}-1} & \text{if } n = 3m-2. \end{cases}$$

Therefore, the series (2) has the same limit as the alternating harmonic series. The alternating harmonic series is often mentioned in calculus: both as an alternating series and as occurring at the right endpoint ($x = 1$) of the interval of convergence for the Maclaurin series of $\ln(x+1)$. This approach requires Abel's theorem on power series [**1**] to show convergence to $\ln(2)$. Some instructors may wish to stop here. However, the section on the series for general k gives a convergence argument that is self-contained for first year calculus.

The general rearrangement for general k. This argument works for $k \geq 2$. The rearrangement of the original series (1) obtained by taking k terms from the harmonic series and subtracting a term from a separate harmonic series gives a rearrangement of the original series (1):

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{k} - 1 + \dots \\ + \frac{1}{k(n-1)+1} + \frac{1}{k(n-1)+2} + \cdots + \frac{1}{k(n-1)+k} - \frac{1}{n} + \dots. \end{aligned} \tag{3}$$

This series converges to $\ln(k)$. The outline of the argument is to first show a subsequence of the sequence of partial sums converges to $\ln(k)$, and then show the series converges to $\ln(k)$.

A subsequence of the sequence of partial sums is a Riemann sum. Write $\{s_n\}_{n=1}^{\infty}$ for the sequence of n-th partial sums of the general series (3). We first show the subsequence $\{s_{n(k+1)}\}_{n=1}^{\infty}$ converges to $\ln(k)$. This partial sum is:

$$\begin{aligned} s_{n(k+1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{k} - 1 + \dots \\ + \frac{1}{k(n-1)+1} + \frac{1}{k(n-1)+2} + \cdots + \frac{1}{k(n-1)+k} - \frac{1}{n}. \end{aligned}$$

Note this partial sum is the difference of its positive terms, which are the first kn terms of the harmonic series, and its negative terms, which are the first n terms of the harmonic series. This difference is:

$$\begin{aligned} &\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{kn}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \cdots + \frac{1}{kn}. \end{aligned}$$

Hence, $s_{n(k+1)} = \frac{1}{n+1} + \cdots + \frac{1}{kn}$. We next show $s_{n(k+1)}$ is a Riemann sum. Begin by rewriting each denominator in the form $n + j$,

$$\begin{aligned} s_{n(k+1)} &= \frac{1}{n+1} + \cdots + \frac{1}{n+(k-1)n} \\ &= \left(\frac{1}{1+\frac{1}{n}} + \cdots + \frac{1}{1+\frac{(k-1)n}{n}}\right)\frac{1}{n}, \quad \text{factoring out } \frac{1}{n}. \end{aligned} \tag{4}$$

Let $f(x) = \frac{1}{1+x}$ on the interval $[0, k-1]$, $\Delta x = \frac{1}{n}$, and $x_i = i\Delta x$ for $i = 1, \dots, (k-1)n$. Equation (4) is then

$$s_{n(k+1)} = \sum_{i=1}^{n(k-1)} f(x_i)\Delta x.$$

As n goes to infinity (so that Δx goes to zero and $n(k-1)$ goes to infinity), the Reimann sum goes to $\int_0^{k-1} \frac{1}{1+x}\,dx = \ln(k)$, i.e.,

$$\lim_{m\to\infty} s_{m(k+1)} = \ln(k).$$

The general k-th series converges to the logarithm of k. We wish to show the general partial sum s_n for the series (3) converges to $\ln(k)$. We look at the series terms after the $m(k+1)$-th term and before the $(m+1)(k+1)$-th term, as we already know $s_{m(k+1)}$ and $s_{(m+1)(k+1)}$:

$$0 < \frac{1}{km+1} + \frac{1}{km+2} + \cdots + \frac{1}{km+j} < \frac{1}{m}$$

for $j = 1, \ldots, k$. So, $s_{m(k+1)} \leq s_{m(k+1)+j} < s_{m(k+1)} + \frac{1}{m}$ for $m > 1$ and $j = 1, \ldots, k$. Hence the partial sums s_n are squeezed to the limit, $\lim_{m\to\infty} s_{m(k+1)} = \ln(k)$, as n goes to ∞.

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Summary. This article gives a nice family of rearrangements of a conditionally convergent series for use in calculus or first analysis classes. In each series, the positive terms and the negative terms both form a harmonic series. For each natural number k, a series is given that converges to the logarithm of k. The series are wonderful to show to any calculus class even if the instructor omits the details. The series or variations are known.

References

1. Apostol, T.M., (1974). *Mathematical Analysis*, 2nd ed. Reading MA: Addison Wesley.